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Short distance operator product expansion of the $1D$, $\mathcal{N} = 4$ Extended $\mathcal{GR}$ super Virasoro algebra by use of coadjoint representations

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Abstract: Using the previous construction of the geometrical representation ($\mathcal{GR}$) of the centerless 1D, $\mathcal{N} = 4$ extended Super Virasoro algebra, we construct the corresponding Short Distance Operation Product Expansions for the deformed version of the algebra. This algebra differs from the regular algebra by the addition of terms containing the Levi-Civita tensor. How this addition changes the super-commutation relations and affects the Short Distance Operation Product Expansions (OPEs) of the associated fields is investigated. The Method of Coadjoint Orbits, which removes the need first to find Lagrangians invariant under the action of the symmetries, is used to calculate the expansions. Finally, future direction of research will be discussed.

Keywords: Extended Supersymmetry, Superspaces.

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Contents

1. Introduction

One of the fundamental mathematical objects of String Theory (ST) is the Virasoro algebra. It is used in the description of simple open/closed strings and is well-developed in Conformal Field Theory (CFT), a primary tool for probing strings. A familiar technique from CFT commonly used in this context is the Operator Product Expansion (OPE) as it is closely related to the calculation of two-point correlation functions which themselves are related to the propagation and interaction of fields represented in ST.

In many discussions, the beginning of such constructions involves first finding an action (containing appropriate fields) that is invariant under a realization of the (super)conformal symmetry group. The solutions of the fields equations of motion are expanded in terms of Fourier series. The Noether charges associated with the generators are, using their expressions in terms of the fields, also then expressed in terms of such Fourier series. Finally OPE's are then calculated. Clearly the role of the action is prominent, both in determing the Noether Charges and the field equations of motion.

Instead the method to be used in this work for calculating these OPE's is the Coadjoint Orbit method developed by Kirillov [1] and built upon the elements of Lie algebras and their realizations. One goes from the closed algebra of operators to elements of a vector space. This vector space is then expanded by the addition of a dual space of covectors and a bilinear metric between the two. These objects are then used to find the coadjoint orbits which can be used in the OPE.

The outline of the paper is as follows. The first section will describe the extended $\mathcal{GR}$ Super Virasoro algebra with a focus on under what conditions does it close and the corresponding number of operators. The next section will explain how the Coadjoint Orbit method is used to generate the Operator Product Expansion. The fourth section shows the calculation of various short distance Operator Product Expansions for the Super Virasoro algebra. Finally, there will be a discussion of some of the implications of the results of the paper.

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Generators	Symmetry	Derivation	No. of generators
P	Translations	$i\,\partial_\tau$	1
Δ	Dilations	$i(\tau\partial_\tau + \frac{1}{2}\zeta^{\rm I}\partial_{\rm I})$	1
K	Special Conformal	$i(\tau^2\partial_\tau + \tau\zeta^{\rm I}\partial_{\rm I})$	1
$Q_{\rm I}$	Supersymmetry	$i(\,\partial_{\rm I} - i\,2\,\zeta_{\rm I}\partial_\tau)$	$4 = [\mathcal{N}]$
$S_{\rm I}$	S-supersymmetry	$i\tau\partial_{\rm I} + 2\tau\zeta_{\rm I}\partial_\tau + \zeta_{\rm I}\zeta^{\rm J}\partial_{\rm J}$	$4 = [\mathcal{N}]$
$T_{\rm I\,J}$	SO($\mathcal{N}$)	$i(\zeta_{\rm I}\,\partial_{\rm J} - \zeta_{\rm J}\,\partial_{\rm I})$	$6 = [\mathcal{N}(\mathcal{N}-1)/2]$

Table 1: Generators and their associated symmetries and derivations.

2. Realization of the 1D, $\mathcal{N} = \triangle$ extended GR Super virasoro algebra

The Super Virasoro algebra can be realized a number of ways including starting from a Lie group and adding a central extension. In the method discussed here, we will not describe the full generation of the algebra from an affine Lie algebra but simply focus of the generators themselves. The Coadjoint Orbit method relies heavily being able to write down the generators and their commutation relations.

The Super Virasoro algebra contains the SO($\mathcal{N}$) algebra using T_{IJ} as generators with translations generated by momentum generators P, the dilations, Δ, special conformal transformations, K, the supersymmetry generators $Q_{\rm I}$ and $S_{\rm I}$. The operators can be represented by derivations of the one dimensional time variable and its derivative, τ and ∂_τ, and the $\mathcal{N} = 4$ superspace variables and their derivatives, $\zeta^{\rm I}$ and $\partial_{\rm I}$. The generators and their corresponding symmetries are listed in table I.

This algebra can be deformed in $\mathcal{N} = 4$ with the addition of a Levi-Civita tensor, $\epsilon_{\rm IJKL}$, and a parameter, ℓ, that measures the deformation. It only affects three of the six operators:

$$S_{\rm I}(\ell) \equiv i\tau\partial_\tau + 2\tau\zeta_{\rm I}\partial_\tau + 2\zeta_{\rm I}\zeta^{\rm J}\partial_{\rm J} + \ell\epsilon_{\rm IJKL}\left(\zeta^{\rm J}\zeta^{\rm K}\partial^{\rm L} - \frac{1}{3!}\zeta^{\rm J}\zeta^{\rm K}\zeta^{\rm L}\partial_\tau\right) \tag{2.1}$$

$$K(\ell) \equiv i\left(\tau^2\partial_\tau + \tau\zeta^{\rm I}\partial_{\rm I} - i\,2\,\ell\epsilon^{\rm IJKL}\left[\frac{1}{4}\zeta_{\rm I}\zeta_{\rm J}\zeta_{\rm K}\partial_{\rm L} + \zeta_{\rm I}\zeta_{\rm J}\zeta_{\rm K}\zeta_{\rm L}\,\partial_\tau\right]\right) \tag{2.2}$$

$$T_{\rm I\,J}(\ell) \equiv i\zeta_{[\rm I}\partial_{\rm J]} - i\ell\epsilon_{\rm IJKL}\zeta_{\rm K}\partial_{\rm L} \tag{2.3}$$

The next step is to recast the previous generators in form in which the relationship to the super Virasoro algebra is more obvious. This is done by choosing the forms

$$L_m \equiv -\left[\tau^{m+1}\partial_\tau + \frac{1}{2}(m+1)\tau^m\zeta\partial_\zeta\right], \qquad H_r \equiv -\left[\tau^{r+1}\partial_\tau + \frac{1}{2}(r+1)\tau^r\zeta\partial_\zeta\right] \tag{2.4}$$

$$F_m \equiv i\tau^{m+\frac{1}{2}}[\partial_\zeta - i\,2\zeta\partial_\tau], \qquad G_r \equiv i\tau^{r+\frac{1}{2}}[\partial_\zeta - i\,2\zeta\partial_\tau] \tag{2.5}$$

where $m \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$. The L and H are the same except L takes integers and H takes half integers. The F and G forms follow the same pattern. H is fermionic and L is bosonic because L exists in the $\mathcal{N} = 0$ case.

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These new generator pairs can be combined using a different notation with simple commutation relations:

$$\begin{pmatrix} L_{\mathcal{A}} \equiv (L_m, H_r) \\ G_{\mathcal{A}} \equiv (F_m, G_r) \end{pmatrix} \rightarrow \begin{pmatrix} [L_{\mathcal{A}}\,,\, L_{\mathcal{B}}\} &=& (\mathcal{A}-\mathcal{B})L_{\mathcal{A}+\mathcal{B}} \\ [G_{\mathcal{A}}\,,\, G_{\mathcal{B}}\} &=& -i\,4\,L_{\mathcal{A}+\mathcal{B}} \\ [L_{\mathcal{A}}\,,\, G_{\mathcal{B}}\} &=& (\tfrac{1}{2}\mathcal{A}-\mathcal{B})\,G_{\mathcal{A}+\mathcal{B}} \end{pmatrix} \tag{2.6}$$

with $\mathcal{A}, \mathcal{B}$ taking values in $\mathbb{Z}$ and $\mathbb{Z}+\frac{1}{2}$. For $\mathcal{N}=1$, this pair of generators is closed under graded commutation. In the 1D $\mathcal{N}=4$ exceptional Super Virasoro algebra, an index I for the supersymmetric levels has to be added and the ℓ-deformed terms must be put in properly, including a ℓ-deformed supersymmetric $T_{\mathrm{I\,J}}(\ell)$ generator. For the 1D $\mathcal{N}=4$ exceptional Super Virasoro algebra, the set of generators $(L_{\mathcal{A}}(\ell),\ G^{\mathrm{I}}_{\mathcal{A}}(\ell),\ T^{\mathrm{I\,J}}_{\mathcal{A}}(\ell))$ closes under graded commutation. These generators are

$$L_{\mathcal{A}} \equiv -\left[\tau^{\mathcal{A}+1}\partial_\tau + \frac{1}{2}(\mathcal{A}+1)\tau^{\mathcal{A}}\zeta^{\mathrm{I}}\partial_{\mathrm{I}}\right] + i\ell\mathcal{A}(\mathcal{A}+1)\tau^{\mathcal{A}-1}[\zeta^{(3)\mathrm{I}}\partial_{\mathrm{I}} + i4\zeta^{(4)}\partial_\tau] \tag{2.7}$$

$$\begin{aligned} G^{\mathrm{I}}_{\mathcal{A}} \equiv\ & \tau^{\mathcal{A}+\frac{1}{2}}[\partial^{\mathrm{I}} - i2\zeta^{\mathrm{I}}\partial_\tau] + 2\left(\mathcal{A}+\frac{1}{2}\right)\tau^{\mathcal{A}-\frac{1}{2}}\zeta^{\mathrm{I}}\zeta^{\mathrm{K}}\partial_{\mathrm{K}} \\ & +\ell\left(\mathcal{A}+\frac{1}{2}\right)\tau^{\mathcal{A}-\frac{1}{2}}[\epsilon^{\mathrm{IJKL}}\zeta_{\mathrm{J}}\zeta_{\mathrm{K}}\partial_{\mathrm{L}} - i4\zeta^{(3)\mathrm{I}}\partial_\tau] + i4\ell\left(\mathcal{A}^2-\frac{1}{4}\right)\tau^{\mathcal{A}-\frac{3}{2}}\zeta^{(4)}\partial^{\mathrm{I}} \end{aligned} \tag{2.8}$$

$$T^{\mathrm{I\,J}}_{\mathcal{A}} \equiv \tau^{\mathcal{A}}[\zeta^{[\mathrm{I}}\partial^{\mathrm{J}]} - \ell\epsilon^{\mathrm{IJKL}}\zeta_{\mathrm{K}}\partial_{\mathrm{L}}] - i2\ell\mathcal{A}\tau^{\mathcal{A}-1}[\zeta^{(3)[\mathrm{I}}\partial^{\mathrm{J}]} - \ell\epsilon^{\mathrm{IJKL}}\zeta^{(3)}_{\mathrm{K}}\partial_{\mathrm{L}}] \tag{2.9}$$

Their supercommutation relations are

$$[L_{\mathcal{A}}\,,\, L_{\mathcal{B}}] = (\mathcal{A}-\mathcal{B})L_{\mathcal{A}+\mathcal{B}} + \frac{1}{8}\,c\,(\mathcal{A}^3-\mathcal{A})\delta_{\mathcal{A}+\mathcal{B},0} \tag{2.10}$$

$$[L_{\mathcal{A}}\,,\, G^{\mathrm{I}}_{\mathcal{B}}] = \left(\frac{\mathcal{A}}{2}-\mathcal{B}\right)G^{\mathrm{I}}_{\mathcal{A}+\mathcal{B}} \tag{2.11}$$

$$[L_{\mathcal{A}}\,,\, T^{\mathrm{I\,J}}_{\mathcal{B}}] = -\mathcal{B}T^{\mathrm{I\,J}}_{\mathcal{A}+\mathcal{B}} \tag{2.12}$$

$$\{G^{\mathrm{I}}_{\mathcal{A}}\,,\, G^{\mathrm{J}}_{\mathcal{B}}\} = -i4\delta^{\mathrm{I\,J}}\,L_{\mathcal{A}+\mathcal{B}} - i\,2\,(\mathcal{A}-\mathcal{B})T^{\mathrm{I\,J}}_{\mathcal{A}+\mathcal{B}} - i\,c\left(\mathcal{A}^2-\frac{1}{4}\right)\delta_{\mathcal{A}+\mathcal{B},0}\delta^{\mathrm{I\,J}} \tag{2.13}$$

$$[T^{\mathrm{I\,J}}_{\mathcal{A}}\,,\, G^{\mathrm{K}}_{\mathcal{B}}] = 2\,(\delta^{\mathrm{J\,K}}\,G^{\mathrm{I}}_{\mathcal{A}+\mathcal{B}} - \delta^{\mathrm{I\,K}}\,G^{\mathrm{J}}_{\mathcal{A}+\mathcal{B}}) \tag{2.14}$$

$$[T^{\mathrm{I\,J}}_{\mathcal{A}}\,,\, T^{\mathrm{K\,L}}_{\mathcal{B}}] = T^{\mathrm{I\,K}}_{\mathcal{A}+\mathcal{B}}\,\delta^{\mathrm{J\,L}} - T^{\mathrm{I\,L}}_{\mathcal{A}+\mathcal{B}}\,\delta^{\mathrm{J\,K}} + T^{\mathrm{J\,L}}_{\mathcal{A}+\mathcal{B}}\,\delta^{\mathrm{I\,K}} - T^{\mathrm{J\,K}}_{\mathcal{A}+\mathcal{B}}\,\delta^{\mathrm{I\,L}} - 2c(\mathcal{A}-\mathcal{B})\,(\delta^{\mathrm{I[K|}}\delta^{\mathrm{J|L]}}) \tag{2.15}$$

A number of interesting points can be found here. In previous papers [2, 3], the non-deformed ($\ell=0$) 1D $\mathcal{N}=4$ $\mathcal{GR}$ Super Virasoro algebra is used to generate OPEs. This algebra is the "large" $\mathcal{N}=4$ algebra which has a 16-dimensional representation. It does not close unless two more *sets* of generators (U's and R's, which are related to the T's,) are added. The $\ell=\pm1$ cases of the ℓ-extended algebra map the generators to a 8-dimensional representation which does not need the other generators to close. This can be seen when instead of using derivations to represent the generators, an appropriately sized Clifford algebra is used [4]. The use of a Clifford algebra may allow more insight into the whole process.

Another point is whether the central extension should be dropped in the equations. From [5], the closure of the algebra is found to be related to the existence of a central extension, specifically the central extension is eliminated for $\mathcal{N} > 2$. Because $\mathcal{N} = 4$ closes also, it is a valid question to ask if a central extension may exist too. The Jacobi Identity on $(G^{\mathrm{I}}_{\mathcal{A}}, U^{\mathrm{IJ}}_{\mathcal{B}}, G^{\mathrm{K}}_{P})$ was used before to answer this question. Because the supercommutators have the same form as the $\mathcal{N} > 2$, it would seem that the answer would be true. But there are no longer $U^{\mathrm{IJ}}_{\mathcal{A}}$ generators in the algebra. The Jacobi identity for the other generators must be analyzed to check if a central extension is allowed. Although this could be addressed now, this question will be revisited later when the Clifford representation of the generators is presented . For now, c will be set to zero.

3. Description of the coadjoint orbit method and operator product expansion

The actual use of the method flows from the following steps:

1. Choose an coadjoint field and an adjoint action on it. This gives the variation of the physical field with respect to some transformation.
2. Calculate the Poisson bracket of the charge generated by the adjoint action on the the physical field. The generating function of the transformation will come from the calculations of the adjoint action on the coadjoint vector done earlier.
3. Compare to the integral form of Poisson bracket. The short distance OPE will be the equivalent expression of the previous step once it has been put in the associated integral form. This will involve the use of delta functions on the space (a line in the 1D case) and its derivatives.

Typically, one needs an action to determine the useful field theory quantities such as correlation functions. However, these quantities are dependent on the symmetries found in the theory and not necessarily obvious in the action. The Coadjoint Orbit method allows for these quantities to be calculated without an action and totally based on the underlying symmetries of the theory being studied.

As an aside, one of the uses of coadjoint orbits is relate the classification of the orbits to the classification of another related mathematical structure. For example, if G is the set of all linear $n \times n$ real invertible matrices, then the classification of coadjoint orbits is equivalent to the classification of matrices up to similarity. The analysis of the coadjoint orbits allows one to classify two dimensional conformal field theories (2D-CFT's).

The Operation Product Expansion (OPE) is an expression of the product of two operators as a sum of singular functions of other operators. This is useful when calculating the product of field operators at the same point. Wilson and Zimmerman [8] have a discussion of the use of OPEs in Quantum Field Theory. In this case, the operators are tensor fields. The general form of an OPE is

$$A(y)B(x) \sim \sum_i C_i(x)(y-x)^{-i} + (\text{non singular terms}) \tag{3.1}$$

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where C_i is a member of a complete set of operators. The non-singular terms are not important because the singular terms determine the properties of the product of operators.

The goal is to express the product of fields that represent the underlying algebra in terms of functions of other fields which represent other elements in the algebra. These products are further related to useful field theory quantities such as propagators and mass terms.

4. Calculation of short distance operator product expansions

The methods used are found in [2, 3, 5, 7]. Applying this process to the algebra of interest, the adjoint vector of the 1D $\mathcal{N} = 4$ $\mathcal{GR}$ SVA is $L = (L_A, G^I_B, T^{\mathrm{JK}}_C)$. The adjoint acting on this gives

$$ad((L_{\mathcal{M}}, G^{\mathrm{K}}_{\mathcal{N}}, T^{\mathrm{LM}}_{\mathcal{P}}))(L_{\mathcal{A}}, G^{\mathrm{I}}_{\mathcal{B}}, T^{\mathrm{JK}}_{CP}) = (L_{\mathcal{M}}, G^{\mathrm{K}}_{\mathcal{N}}, T^{\mathrm{LM}}_{\mathcal{P}}) * (L_{\mathcal{A}}, G^{\mathrm{I}}_{\mathcal{B}}, T^{\mathrm{IJ}}_{P}) \tag{4.1}$$
$$= (L_{\mathcal{Q},\mathrm{new}}, G^{\mathrm{H}}_{\mathcal{R},\mathrm{new}}, T^{\mathrm{FG}}_{\mathcal{S},\mathrm{new}}) \tag{4.2}$$

The coadjoint element is $\tilde{L} = (\tilde{L}_{\mathcal{A}}, \tilde{G}^{\mathrm{I}}_{\mathcal{B}}, \tilde{T}^{\mathrm{JK}}_P)$ and correspondingly gives

$$ad((L_{\mathcal{M}}\,,\, G^{\mathrm{K}}_{\mathcal{N}}\,,\, T^{\mathrm{LM}}_{\mathcal{P}}))\,(\tilde{L}_{\mathcal{A}}, \tilde{G}^{\mathrm{I}}_{\mathcal{B}}, \tilde{T}^{\mathrm{JK}}_P) = (L_{\mathcal{M}}, G^{\mathrm{K}}_{\mathcal{N}}, T^{\mathrm{LM}}_{\mathcal{P}}) * (\tilde{L}_{\mathcal{A}}, \tilde{G}^{\mathrm{I}}_{\mathcal{B}}, \tilde{T}^{\mathrm{JK}}_P) \tag{4.3}$$
$$= (\tilde{L}_{\mathcal{Q},\mathrm{new}}, \tilde{G}^{\mathrm{H}}_{\mathcal{R},\mathrm{new}}, \tilde{T}^{\mathrm{FG}}_{\mathrm{S},\mathrm{new}}) \tag{4.4}$$

and the inner product is

$$\langle(\tilde{L}_M, \tilde{G}^K_N, \tilde{T}^{LM}_P)|(L_A, G^I_B, T^{JK}_C)\rangle = \delta_{M,A} + \delta_{N,B}\delta^I_K + \delta_{P,C}\delta^{JK}_{LM} \tag{4.5}$$

To calculate the OPEs, one needs the expression of $\delta_L \tilde{L} = L * \tilde{L}$ where L is an adjoint vector and $\tilde{L}$ is a coadjoint vector. Using the fact that $\langle\tilde{L}|L\rangle$ is an invariant and $L * \tilde{L}$ can be calculated from $\langle L' * \tilde{L}|L\rangle$, one can use the Leibnitz rule on the invariant form and get

$$\langle L * \tilde{L}|L\rangle = -\langle\tilde{L}|L * L\rangle \tag{4.6}$$

Since L and $\tilde{L}$ are made up of components (L, G, T), it is easier to calculate pairs of adjoint elements acting on coadjoint elements. This reduces the number of calculations greatly. The list of adjoint/coadjoint pairs are

$$\begin{aligned}
\delta\tilde{L} &= L * \tilde{L} + G * \tilde{G} + T * \tilde{T} \\
\delta\tilde{G} &= L * \tilde{G} + G * \tilde{L} + G * \tilde{T} + T * \tilde{G} \\
\delta\tilde{T} &= L * \tilde{T} + G * \tilde{G} + T * \tilde{T}
\end{aligned}$$

This checks against the calculations from [3]. Note that there is no $\tilde{T} * L$ term in the list of changes to the coadjoint vector.

Using a realization of the algebra as tensor fields, the adjoint representation elements are $F = (\eta, \chi^I, t^{\mathrm{RS}})$, which are general elements of the Virasoro, Kac-Moody, and so(4) algebras respectively. The coadjoint fields are $B = (D, \psi^I, A^{\mathrm{RS}})$, a rank two pseudo tensor, a set of 4 spin-3/2 fields, and the 6 so(4) gauge fields.

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The coadjoint action can be seen as generating the changes in the fields. It acts as

$$F * \tilde{B} = \delta_F \tilde{B} = (\eta, \chi^{\mathrm{J}}, t^{\mathrm{K\,L}}) * (D, \psi^{\mathrm{I}}, A^{\mathrm{J\,K}}) = (\delta D, \delta\psi^{\mathrm{I}}, \delta A^{\mathrm{J\,K}}). \tag{4.7}$$

There are three charges, one for each adjoint element/operator:

$$\begin{aligned}
L_A \to \eta \to Q_\eta &= \int dx\, G_a \eta^a && (4.8)\\
G_A^I \to \chi^{\mathrm{I}} \to Q_{\chi^{\mathrm{I}}} &= \int dx\, G_a (\chi^{\mathrm{I}})^a && (4.9)\\
T_A^{IJ} \to t^{\mathrm{I\,J}} \to Q_{t^{\mathrm{I\,J}}} &= \int dx\, G_a (t^{\mathrm{I\,J}})^a && (4.10)
\end{aligned}$$

Choosing $L * \tilde{L}$ as an example, the physical field representation is used:

$$\begin{aligned}
L * \tilde{L} &\to \delta_\eta D && (4.11)\\
L_\eta * \tilde{L}_D \to \tilde{L}_{\tilde{D}} : \tilde{D} &= -D'\eta - 2D\eta' && (4.12)\\
\delta_\eta D &= \tilde{D} && (4.13)\\
\delta_\eta D &= -\{Q_\eta, D\} = \int dy\, \eta(y)(D(y)D(x)) && (4.14)\\
Q_\eta &= \int dx\, G^a \eta_a = \int dx(-D'\eta - 2D\eta')\eta_a && (4.15)\\
\{Q_\eta, D\} &= \int dy(-D'(x)\eta(y) - 2D(x)\eta'(y)). && (4.16)
\end{aligned}$$

Using the 1D formula for the delta function,

$$\delta(y - x) = \frac{1}{2\pi i(y - x)} \tag{4.17}$$

and integration by parts to separate out $\eta(x)$ terms,

$$\begin{aligned}
\{Q_\eta, D\} &= \int dy \underbrace{\left(\partial_x D(x)\frac{-1}{2\pi i(y-x)} + D(x)\frac{-1}{\pi i(y-x)^2}\right)} \eta(x)\\
&= \int dy \qquad\qquad [D(y)D(x)] \qquad\qquad \eta(x)
\end{aligned} \tag{4.18}$$

Thus by taking pairs of individual adjoint elements acting on individual coadjoint elements, the OPE's can found.

1. $D(y)O(x)$

$$\begin{aligned}
L_\eta * \tilde{L}_D = \ \tilde{L}_{\tilde{D}} \to \tilde{D} \ &= -D'\eta - 2D\eta' && (4.19)\\
L_\eta * \tilde{G}^{\bar{Q}}_{\psi^{\bar{Q}}} = \tilde{G}^{\bar{Q}}_{\tilde{\psi}^{\bar{Q}}} \to \tilde{\psi}^{\bar{Q}} \ &= -\left(\frac{3}{2}\eta'\psi^{\bar{Q}} - \eta(\psi^{\bar{Q}'})\right) && (4.20)\\
L_\eta * \tilde{T}^{\bar{R}\bar{S}}_{A^{\bar{I}\bar{J}}} = \tilde{T}^{\bar{R}\bar{S}}_{\tilde{A}^{\bar{I}\bar{J}}} \to \tilde{A}^{\bar{I}\bar{J}} \ &= -(A^{RS})'\eta - \eta' A^{RS} && (4.21)
\end{aligned}$$

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These expressions yield the following OPEs:

$$\begin{aligned} D(y)D(x) &= \frac{-1}{\pi i(y-x)^2}D(x) - \frac{1}{2\pi(y-x)}\partial_x D(x) && (4.22)\\ D(y)\psi^Q(x) &= \frac{-3}{4\pi i(y-x)^2}\psi^Q(x) - \frac{1}{2\pi i(y-x)}\partial_x \psi^Q(x) && (4.23)\\ D(y)A^{RS}(x) &= \frac{-1}{2\pi i(y-x)^2}A^{RS}(x) - \frac{1}{2\pi i(y-x)}\partial_x A^{RS}(x) && (4.24) \end{aligned}$$

2. $\psi(y)O(x)$

$$\begin{aligned} G^I_{\chi^I} * \tilde{L}_D = 4i\tilde{G}^I_{\tilde{\chi}^I} \rightarrow \quad \tilde{\chi}^I &= -\chi^I D && (4.25)\\ G^I_{\chi^I} * \tilde{G}^{\bar{Q}}_{\psi^{\bar{Q}}} = \frac{\delta^{I\bar{Q}}}{2}\tilde{L}_{\tilde{D}} + \tilde{T}^{I\tilde{Q}}_{\tilde{A}^{I\tilde{Q}}} \rightarrow \quad \tilde{D} &= \left[(\psi^{\bar{Q}})'\psi^I - 3(\psi^I)'\psi^Q\right] \\ \tilde{A}^{I\tilde{Q}} &= 2(\chi^I\psi^{\tilde{Q}} - \chi^{\tilde{Q}}\psi^I) && (4.26)\\ G^I_{\chi^I} * \tilde{T}^{\bar{R}\bar{S}}_{\tau^{\bar{R}\bar{S}}} = \delta^{[RS][IQ]}\tilde{G}^Q_{\tilde{\psi}^Q} \rightarrow \quad \psi^{\bar{Q}} &= 2(\chi^I)'t^{RS} + \chi^I(t^{RS})' && (4.27) \end{aligned}$$

The OPEs are

$$\begin{aligned} \psi^I(y)D(x) &= \frac{-3}{4\pi i(y-x)^2}\psi^I(x) - \frac{i}{4\pi(y-x)}\partial_x\psi^I(x) && (4.28)\\ \psi^I(y)\psi^Q(x) &= \frac{-4i}{(y-x)}\delta^{IQ}D(x) && (4.29)\\ \psi^A(y)A^{RS}(x) &= \frac{\pi}{i(y-x)}(\delta^{AR}\delta^{LS} - \delta^{AS}\delta^{LR})\psi^L(x) && (4.30) \end{aligned}$$

3. $A(y)O(x)$

$$\begin{aligned} T^{IJ}_{t^{IJ}} * \tilde{G}^{\bar{Q}}_{\psi^{\bar{Q}}} &= 2\delta^{QI}G^J_{\tilde{\phi}^J} - 2\delta^{QJ}G^I_{\tilde{\phi}^I} \rightarrow \psi^{\bar{Q}} = t^{IJ}\psi^Q && (4.31)\\ T^{IJ}_{t^{IJ}} * \tilde{T}^{\bar{R}\bar{S}}_{\tau^{\bar{R}\bar{S}}} &= -\delta^{[\bar{R}\bar{S}]}(\delta^{JK}_{RS} + \delta^{KJ}_{RS})\tilde{T}^{\bar{R}\bar{S}}_{(t^{JK})'\tau^{\bar{R}\bar{S}}} - \tilde{L}_{\tilde{D}}\delta^{[\bar{R}\bar{S}][JK]} \rightarrow D = (t^{JK})'\tau^{\bar{R}\bar{S}} && (4.32) \end{aligned}$$

Note that there is no $T * \tilde{L}$ term. However the $A^{JK}(y)D(x)$ and $A^{JK}(y)A^{RS}(x)$ terms are generated from the $T * \tilde{T}$ action. The OPE that follow are

$$\begin{aligned} A^{JK}(y)D(x) &= \frac{1}{4\pi i(y-x)^2}(\delta^{RS}\delta^{JK} - \delta^{RK}\delta^{LS})A_{RS}(x) && (4.33)\\ A^{AB}(y)\psi^C(X) &= \frac{-1}{\pi i(y-x)}(\delta^{AC}\psi^B(x) - \delta^{AB}\psi^C(x)) && (4.34)\\ A^{JK}(y)A^{RS}(x) &= \frac{1}{4\pi i(y-x)}\delta^{JKRS}_{AB}A^{AB}(x) && (4.35) \end{aligned}$$

In the non-extended version of the algebra [2, 3, 5, 7], there are extra generators that must be added to close the algebra. When the Coadjoint Orbit method is applied, these extra generators correspond to fields and have their own OPEs. The fields ω and ρ, which correspond to the U and R operators respectively, have 44 and 11 independent components.

The spin of the fields are varied, either being 0 or $\frac{1}{2}$ depending on the structure of the individual operator. This also true for the general extended $\ell \neq \pm 1$ case. However, the $\ell = \pm 1$ case does not have these fields or their OPEs. Thus there is no difference between the regular (ℓ=0) and extended ($\ell \neq 0$) cases except when $\ell = \pm 1$. These cases reduce the number of operators and fields necessary to fully describe the theory.

5. Discussions, interpretations, and conclusions

There are a number of interesting ideas and directions that this work has brought up:

- Clifford Algebra Representation: Hasiewicz, Thielemans, and Troost [4] have shown that superconformal Lie superalgebras contain a Clifford algebra structure in them. By exploiting this structure, new information can be gained by the implications of how the Clifford algebra exists in the larger structure. With the algebra elements written as elements of a Clifford algebra, all of the previous work can be double checked and reanalyzed in a different context. The benefit of going to a Clifford algebra representation is that the Clifford algebras are well-known and well-understood. In [4], there is some discussion about what this would entail and will be investigated for future research.

- Coadjoint Orbit Method: The Coadjoint Orbit method has a clear mathematical basis underlying it. There exists a relationship between the equivalence class of linear functions on a Lie group (trajectories) and a natural sympletic structure on the relevant manifold (phase space). The connection between the two seems more obvious in terms of Clifford algebras, which has a foot in both worlds. It may be that a simpler explanation can be found by exploring this direction with the first step going from the Clifford algebras to the underlying Spin groups and algebras which are closer related to Lie groups.

- Higher-point functions (3-point and 4-point correlators): The methods of this paper describe using any representation of symmetry generators to develop OPEs describing two-point correlation functions. In [8], there is a way to extend this methodology to higher point functions. Thus, it may be possible to totally "skip" Hamiltionian and Langrangian and just calculate correlation functions from symmetries. Skipping that step, however, does not absolve one from still figuring out the dynamics of the theory, which are contained in the propagation and interaction terms calculated from the OPEs. Also, one must deal with the difficulties of the constraints on 3-point and 4-point functions from Conformal Field Theory.

- Since the Virasoro and Kac-Moody algebras are Lie algebras, they have interpretations as manifolds. What does the central extension mean in terms of manifolds? A central extension in group representation terms means that there are operators (or combinations of operators) that exist in the center of the group besides the typical identity element. The formal name for this concept is an ideal, a subgroup that maps products between members inside and outside the subgroup into the subgroup. In this case, it represents that elements in the group can be pulled back to "another origin". The interpretation of the central extension should be important for any work involving Geometrical Representation theory.

- In [4], they discussed the non-existence of a description of superconformal Lie superalgebras with dim $W > 4$. There were a number of restrictions to this statement but they discuss $\mathcal{N} > 4$ superconformal superalgebras that were not Lie superalgebras. Further research into this area could provide a possible generalization of supersymmetry algebras.

Our use of super vector fields in order to realize the symmety generators in a geometrical manner also points in one other direction. Since there is no metric defined on a Salam-Strathdee superspace, the conventional and familiar role of the metric (or a putative super-metric) is taken over by super-frame fields or super vielbeins. Thus a definition of Killing super-vectors must rely on a super vielbein. As such there is a superspace geometry that is naturally associated with the vector fields (realizing the symmetry). This geometry is the conventional one of a flat Salam-Strathdee superspace. This raises a question. One can imagine a super vielbein that does *not* describe a flat Salam-Strathdee superspace but one with a non-trivial topology. If it possesses a related set of Killing super vectors. In principle it should be possible to derive short distance expansions in this case.

In conclusion, the short distance OPEs for the extended 1D $\mathcal{N} = 4$ Super Virasoro algebra was calculated and found to be exactly of the same form of the 1D $\mathcal{N} = 2$ case. Further investigation showed the full relationship between the "large" and "small" $\mathcal{N} = 4$ algebras and the deeper relationship between the two through the Clifford algebra. Let us end by noting that the relation to Clifford algebras also suggest that 'Garden Algebras' defined in [9] seem likely to provide a starting point for some OPE's.

> "*No human investigation can be called real science if it cannot be demonstrated mathematically.*"
> — Leonardo da Vinci

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